

Interval colorings of edges of a multigraph

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A translation from Russian of the work of A.S. Asratian and R.R. Kamalian "Interval colorings of edges of a multigraph", Applied Mathematics 5, Yerevan State University, 1987, pp. 25–34.

Let $G = (V(G), E(G))$ be a multigraph. The degree of a vertex x in G is denoted by $d(x)$, the greatest degree of a vertex – by $\Delta(G)$, the chromatic index of G – by $\chi'(G)$. Let $R \subseteq V(G)$.

An interval (respectively, continuous) on R t -coloring of a multigraph G is a proper coloring of edges of G with the colors $1, 2, \dots, t$, in which each color is used at least for one edge, and the edges incident with each vertex $x \in R$ are colored by $d(x)$ consecutive colors (respectively, by the colors $1, 2, \dots, d(x)$).

In this paper the problems of existence and construction of interval or continuous on R colorings of G are investigated. Problems of such kind appear in construction of timetablings without "windows". Some properties of interval or continuous on $V(G)$ colorings were obtained in [1, 2]. Necessary and sufficient conditions of the existence of a continuous on $V(G)$ $\Delta(G)$ -coloring in the case when G is a tree are obtained in [3]. All non-defined concepts can be found in [4, 5].

Let \mathfrak{N}_t be the set of multigraphs G , for which there exists an interval on $V(G)$ t -coloring, and $\mathfrak{N} = \bigcup_{t \geq 1} \mathfrak{N}_t$. For every $G \in \mathfrak{N}$, let us denote by $w(G)$ and $W(G)$, respectively, the least and the greatest t , for which there exists an interval on $V(G)$ t -coloring of G . Evidently, $\Delta(G) \leq \chi'(G) \leq w(G) \leq W(G)$.

Proposition 1. *If $G \in \mathfrak{N}$ then $\chi'(G) = \Delta(G)$.*

Proof. Let us consider an interval on $V(G)$ $w(G)$ -coloring of the multigraph G . If $w(G) = \Delta(G)$ then $\chi'(G) = \Delta(G)$. Assume that $w(G) > \Delta(G)$. Let us define the sets $T(1), \dots, T(\Delta(G))$, where $T(j) = \{i / i \equiv j \pmod{\Delta(G)}, 1 \leq i \leq w(G)\}$, $j = 1, \dots, \Delta(G)$. Let E_j be the subset of edges of G which are colored by colors from the set $T(j)$, $j = 1, \dots, \Delta(G)$. Clearly, E_j is a matching. For each $j \in \{1, \dots, \Delta(G)\}$, let us color the edges of E_j by the color j . We shall obtain a proper coloring of edges of G with $\Delta(G)$ colors. Hence, $\chi'(G) = \Delta(G)$.

The Proposition is proved.

Proposition 2. *Let G be a regular multigraph.*

- a) $G \in \mathfrak{N}$ iff $\chi'(G) = \Delta(G)$.
- b) If $G \in \mathfrak{N}$ and $\Delta(G) \leq t \leq W(G)$ then $G \in \mathfrak{N}_t$.

Proof. The proposition (a) follows from the proposition 1. The proposition (b) holds, since if $t > w(G)$ then an interval on $V(G)$ $(t - 1)$ -coloring can be obtained from an interval on $V(G)$ t -coloring by recoloring with the color $t - \Delta(G)$ all edges colored by t .

The Proposition is proved.

It is proved in [6] that for a regular graph G , the problem of deciding whether $\chi'(G) = \Delta(G)$ or $\chi'(G) \neq \Delta(G)$ is NP -complete by R. Karp [7]. It follows from here and from the proposition 2

that for a regular graph G , the problem of determining whether $G \in \mathfrak{N}$ or $G \notin \mathfrak{N}$ is NP -complete by R. Karp.

Lemma 1. *Let G be a connected multigraph with a proper edge coloring with the colors $1, \dots, t$, and the edges incident with each vertex $x \in V(G)$ are colored by $d(x)$ consecutive colors. Then $G \in \mathfrak{N}$.*

Proof. Let $\alpha(e)$ be the color of the edge $e \in E(G)$. Without loss of generality, we assume that $\min_e \alpha(e) = 1$, $\max_e \alpha(e) = t$. For the proof of the lemma it is suffice to show, that if $t \geq 3$, then each color r , $1 < r < t$, is used for at least one edge. Since G is connected, then there exists a simple path $P = (x_0, e_1, x_1, \dots, x_{k-1}, e_k, x_k)$ in it, where $e_i = (x_{i-1}, x_i)$, $i = 1, \dots, k$ and $\alpha(e_1) = t$, $\alpha(e_k) = 1$. If $\alpha(e_i) \neq r$, $i = 1, \dots, k$, let us consider in P the vertex x_{i_0} with the greatest index, satisfying the inequality $\alpha(e_{i_0}) > r$. Then $\alpha(e_{1+i_0}) < r$. It follows from the condition of the lemma that there is an edge incident with the vertex x_{i_0} colored by the color r .

The Lemma is proved.

Theorem 1. *Let G be a connected graph without triangles. If $G \in \mathfrak{N}$ then $W(G) \leq |V(G)| - 1$.*

Proof by contrary. Assume that there exist connected graphs H in \mathfrak{N} without triangles with $W(H) \geq |V(H)|$. Let us choose among them a graph G with the least number of edges. Clearly, $|E(G)| > 1$. Consider an interval on $V(G)$ $W(G)$ -coloring of G . The color of an edge e is denoted by $\alpha(e)$, the set $\{v \in V(G) / (u, v) \in E(G)\} -$ by $I(u)$. Let \mathfrak{M} be the set of all simple paths with the initial edge colored by the color $W(G)$ and the final edge colored by the color 1. For each $P \in \mathfrak{M}$ with the sequence e_1, \dots, e_t of edges, $t \geq 2$, let us set in correspondence the sequence $\alpha(P) = (\alpha(e_1), \dots, \alpha(e_t))$ of colors. Let us show that there is a path P_0 in \mathfrak{M} for which $\alpha(P_0)$ is decreasing.

Let $\alpha(e') = W(G)$, $e' = (x_0, x_1)$, and $d(x_1) \geq d(x_0)$. Since $|E(G)| > 1$, then $d(x_1) \geq 2$. Let us construct the sequence X of vertices as follows:

Step 1. $X := \{x_0, x_1\}$.

Step 2. Let x_i be the last vertex in the sequence X . If $I(x_i) \setminus X = \emptyset$ or $\alpha(x_i, y) > \alpha(x_{i-1}, x_i)$ for each $y \in I(x_i) \setminus X$ then the construction of X is completed. Otherwise let us choose from $I(x_i) \setminus X$ the vertex x_{i+1} , for which $\alpha(x_i, x_{i+1}) = \min \alpha(x_i, y)$, where the minimum is taken on all $y \in I(x_i) \setminus X$. Let us bring x_{i+1} in X and repeat the step 2.

Suppose that X is constructed and $X = \{x_0, x_1, \dots, x_k\}$. Clearly, X defines a simple path $P_0 = (x_0, e_1, x_1, \dots, x_{k-1}, e_k, x_k)$, where $e_i = (x_{i-1}, x_i)$, $i = 1, \dots, k$. Let us show that $\alpha(e_k) = 1$.

Suppose $1 < \alpha(e_k) < W(G)$. Let us define a graph H as follows:

$$H = \begin{cases} G - x_k, & \text{if } d(x_k) = 1 \\ G - e_k, & \text{if } d(x_k) \geq 2. \end{cases}$$

Let us show that H is connected. Assume the contrary. Then $H = G - e_k$. Let H_1, H_2 be the connected components of H , $x_{k-1} \in V(H_1)$, $x_k \in V(H_2)$, and G_1, G_2 be the subgraphs of G induced, respectively, by the subsets $V(H_1) \cup \{x_k\}$ and $V(H_2) \cup \{x_{k-1}\}$. The coloring of the graph G induces the coloring of G_i satisfying the conditions of the lemma 1. Therefore $G_i \in \mathfrak{N}$, and, since $|E(G_i)| < |E(G)|$ then $W(G_i) \leq |V(G_i)| - 1$, $i = 1, 2$. It is not difficult to check that $W(G) \leq W(G_1) + W(G_2) - 1$. From here we obtain the inequality $W(G) \leq |V(G)| - 1$, which contradicts the choice of G . Therefore H is connected.

It follows from the lemma 1 that the coloring of edges of H induced by the coloring of G is an interval on $V(H)$ $W(G)$ -coloring. Then $W(H) \geq W(G) \geq |V(G)| \geq |V(H)|$. The obtained inequality contradicts the choice of G , because $|E(H)| < |E(G)|$. Consequently, $\alpha(e_k) = 1$.

Hence, we have constructed the path $P_0 \in \mathfrak{M}$ for which the sequence $\alpha(P_0)$ is decreasing.

Let us denote by ϑ the set of all shortest paths P from \mathfrak{M} , for which the sequence $\alpha(P)$ is decreasing. Let k be the length of paths in ϑ .

Let us define the sets $\vartheta_1, \dots, \vartheta_k$: $\vartheta_1 = \vartheta$, and ϑ_i is the subset of paths from ϑ_{i-1} with the greatest color of the i -th edge, $i = 2, \dots, k$.

Let us choose from ϑ_k some path $P_1 = (x_0, e_1, x_1, \dots, x_{k-1}, e_k, x_k)$. Let $A(i) = \{y \in I(x_i) / \alpha(e_{i+1}) < \alpha(x_i, y) < \alpha(e_i)\}$. Clearly, $|A(i)| = \alpha(e_i) - \alpha(e_{i+1}) - 1$, $i = 1, \dots, k-1$. Let us show that $A(i) \cap \{x_0, x_1, \dots, x_k\} = \emptyset$, $i = 1, \dots, k-1$. Suppose that there exist such i_0, j_0 , that either $x_{i_0} \in A(j_0)$ or $x_{j_0} \in A(i_0)$. Let us define the path P as follows. If $i_0 \neq 0$, $j_0 \neq k$, then $P = (x_0, e_1, x_1, \dots, x_{i_0}, (x_{i_0}, x_{j_0}), x_{j_0}, \dots, x_k)$. If $i_0 = 0$, then $P = (x_1, e_1, x_0, (x_0, x_{j_0}), x_{j_0}, \dots, x_k)$. If $j_0 = k$, then $P = (x_0, e_1, x_1, \dots, x_{i_0}, (x_{i_0}, x_k), x_k, e_k, x_{k-1})$. In all three cases the sequence $\alpha(P)$ is decreasing, and the length of P is less than the length of P_1 , which contradicts the choice of P_1 . Therefore

$$A(i) \cap \{x_0, x_1, \dots, x_k\} = \emptyset, \quad i = 1, \dots, k-1 \quad (1)$$

Let us show that $A(i) \cap A(j) = \emptyset$, $1 \leq i < j \leq k-1$. Suppose that there exist such i_0, j_0 , for which $1 \leq i_0 < j_0 \leq k-1$ and $A(i_0) \cap A(j_0) \neq \emptyset$.

Since there is no triangle in G then $j_0 - i_0 \geq 2$. Let $v \in A(i_0) \cap A(j_0)$. Let us consider the path $P = (x_0, e_1, x_1, \dots, x_{i_0}, (x_{i_0}, v), v, (v, x_{j_0}), x_{j_0}, \dots, x_{k-1}, e_k, x_k)$. Clearly, the sequence $\alpha(P)$ is decreasing. If $j_0 - i_0 \geq 3$ then the length of P is less than the length of P_1 , and if $j_0 - i_0 = 2$ then $\alpha(x_{i_0}, v) > \alpha(e_{1+i_0})$. In both cases it contradicts the choice of P_1 . Therefore

$$A(i) \cap A(j) = \emptyset, \quad 1 \leq i < j \leq k-1 \quad (2)$$

From (1) and (2), it follows that

$$\begin{aligned} |V(G)| &\geq \left| \bigcup_{i=1}^{k-1} A(i) \right| + k + 1 = k + 1 + \sum_{i=1}^{k-1} |A(i)| = k + 1 + \sum_{i=1}^{k-1} (\alpha(e_i) - \alpha(e_{i+1}) - 1) = \\ &= k + 1 + W(G) - 1 - (k-1) = 1 + W(G). \end{aligned}$$

It contradicts the choice of G .

The Theorem is proved.

Corollary 1. *If G is a connected bipartite graph, and $G \in \mathfrak{N}$, then $W(G) \leq |V(G)| - 1$.*

Let $G = (V_1(G), V_2(G), E(G))$ be a bipartite multigraph. Let us denote by $w_1(G)$ and $W_1(G)$, respectively, the least and the greatest t , for which there exists an interval on $V_1(G)$ t -coloring of G . Evidently, $W_1(G) = |E(G)|$.

Theorem 2. *For any t , $w_1(G) \leq t \leq W_1(G)$, there exists an interval on $V_1(G)$ t -coloring of the multigraph G .*

Proof by induction on $|V_1(G)|$.

If $|V_1(G)| = 1$ then the proposition of the theorem is true. Suppose that the proposition of the theorem is true for all G' with $|V_1(G')| = p$. Suppose that $|V_1(G)| = p + 1$, and assume there exists an interval on $V_1(G)$ t -coloring of G , $w_1(G) \leq t < W_1(G)$. Among vertices of $V_1(G)$ which are incident with edges colored by the color t , let us choose a vertex x_1 with the smallest degree. There is an edge e_1 colored by the color $t + 1 - d(x_1)$ which is incident with the vertex x_1 .

1) If there exists an edge different from e_1 and colored by the color $t + 1 - d(x_1)$, then, by recoloring e_1 with the color $t + 1$ we shall obtain an interval on $V_1(G)$ $(t + 1)$ -coloring of G .

2) Let e_1 be the unique edge colored by the color $t + 1 - d(x_1)$, and s be the maximum color which is used for more than one edge. Clearly, $1 \leq s < t < |E(G)|$.

2a) Let $t + 1 - d(x_1) < s < t$. Let us recolor each edge with the color i , where $i = t + 1 - d(x_1), \dots, s$, by the color $i + t - s$, and let us recolor each edge with the color i , where $i = s + 1, \dots, t$, by the color $(i + t - s)(\text{mod } t) + t - d(x_1)$. In the obtained interval on $V_1(G)$ t -coloring, among that vertices from $V_1(G)$ which are incident with edges colored by t (there are more than 1 such edges), we shall choose a vertex x_2 with the smallest degree and recolor the incident with it edge with the color $t + 1 - d(x_2)$ by the color $t + 1$. We shall obtain an interval on $V_1(G)$ $(t + 1)$ -coloring of G .

2b) Let $1 \leq s < t + 1 - d(x_1)$. Removing x_1 from G , we shall obtain a multigraph G' with an interval on $V_1(G')$ $(t - d(x_1))$ -coloring. Clearly, $|E(G')| = |E(G)| - d(x_1)$ and $t - d(x_1) < |E(G')| = W_1(G')$. By the assumption of induction there exists an interval on $V_1(G')$ $(t + 1 - d(x_1))$ -coloring of G' . We shall color the edges incident with the vertex x_1 by the colors $t + 2 - d(x_1), \dots, t + 1$ and obtain an interval on $V_1(G)$ $(t + 1)$ -coloring of G .

The Theorem is proved.

In the work [8] in terms of timetables the NP -completeness was proved for the problem of a 3-coloring of a bipartite graph with preassignments in one part. A bipartite graph $H = (V_1(H), V_2(H), E(H))$ with $\Delta(H) = 3$ is given, where the set $V_1(H)$ contains no pendent vertex, and, for each $x \in V_1(H)$, a set $T(x)$ is preassigned, $T(x) \subseteq \{1, 2, 3\}$, $|T(x)| = d_H(x)$. The required is to determine does there exist a proper coloring of edges of H with the colors 1, 2, 3, at which the edges incident with each vertex $x \in V_1(H)$ are colored by colors from the set $T(x)$.

Theorem 3. *For a bipartite multigraph with the greatest degree 3 of a vertex, the problem of deciding whether a 3-coloring, continuous on one part, exists or not, is NP -complete.*

Proof. Let H' be a graph isomorphic to the graph H , $V(H) \cap V(H') = \emptyset$, and to each vertex $y \in V(H)$ a vertex $y' \in V(H')$ corresponds. Let us construct a bipartite multigraph G_1 as follows.

For each $y \in V_2(H)$, connect the vertices y and y' with one edge if $d_H(y) = 2$, and with two parallel edges if $d_H(y) = 1$. Clearly, $\Delta(G_1) = 3$.

Set

$$\begin{aligned} T(y') &:= T(y), T(y) := T(y) \text{ for each } y \in V_1(H), \\ T(y') &:= \{1, 2, 3\}, T(y) := \{1, 2, 3\} \text{ for each } y \in V_2(H). \end{aligned}$$

Let $V_{ij} = \{y \in V(G_1) / T(y) = \{i, j\}\}$, $1 \leq i < j \leq 3$.

Let us define a bipartite multigraph G as follows:

$$\begin{aligned} V(G) &= V(G_1) \cup \{x_1 / x \in V_{23}\} \cup \{x_1, x_2 / x \in V_{13}\}, \\ E(G) &= E(G_1) \cup \{(x, x_1) / x \in V_{23}\} \cup \{(x, x_1), (x_1, x_2) / x \in V_{13}\}. \end{aligned}$$

Clearly, a 3-coloring of H with preassignments in $V_1(H)$ exists if and only if a 3-coloring of edges of G_1 exists at which the edges incident with each vertex $x \in V(G_1)$ are colored by colors from the set $T(x)$. Such coloring of edges of G_1 exists if and only if a 3-coloring of G continuous on $V(G)$ exists. It is not difficult to check that the collection of degrees of vertices of $V_1(G)$ coincides with the collection of degrees of vertices of $V_2(G)$. Therefore a continuous on $V(G)$ 3-coloring of G exists if and only if a continuous on $V_1(G)$ 3-coloring of G exists.

The Theorem is proved.

Theorem 4. *Let $G = (V_1(G), V_2(G), E(G))$ be a bipartite multigraph. If for each edge (x, y) , where $x \in V_1(G)$, the condition $d(x) \geq d(y)$ holds, then G has a continuous on $V_1(G)$ $\Delta(G)$ -coloring.*

Proof. Let $V_1(G) = \{x_1, \dots, x_p\}$, $d(x_1) \geq \dots \geq d(x_p)$, and already a proper coloring of edges incident with the vertices x_1, \dots, x_n ($n \geq 1$) is constructed so that the edges incident with the vertex x_i are colored by the colors $1, \dots, d(x_i)$, $i = 1, \dots, n$. If $n < p$ and with the vertex x_{n+1} the edges $(x_{n+1}, y(1)), \dots, (x_{n+1}, y(d(x_{n+1})))$ are incident, then, sequentially for each $j = 1, \dots, d(x_{n+1})$ do as follows. If the color j is absent in the vertex $y(j)$, then we shall color the edge $(x_{n+1}, y(j))$ by the color j . Otherwise a color k is absent in $y(j)$, $1 \leq k \leq d(x_{n+1})$. We shall recolor the longest path consisting of edges colored by j and k with the initial vertex $y(j)$ and we shall color the edge $(x_{n+1}, y(j))$ by the color j .

The Theorem is proved.

Corollary 2. *If*

$$\min_{x \in V_1(G)} d(x) \geq \max_{y \in V_2(G)} d(y),$$

then G has a continuous on $V_1(G)$ $\Delta(G)$ -coloring.

Proposition 3. *The problem of deciding whether a proper coloring of edges of a bipartite multigraph with the fixed number of edges of each color exists is NP-complete.*

Proof. Let $G = (V_1(G), V_2(G), E(G))$ be a bipartite multigraph with $\Delta(G) = 3$. Set $n_i = |\{x / x \in V_1(G), d(x) \geq i\}|$, $i = 1, 2, 3$. Clearly, a continuous on $V_1(G)$ 3-coloring of G exists if and only if there exists a proper coloring of edges of G with the colors 1, 2, 3, at which by each color i n_i edges are colored, $i = 1, 2, 3$. Therefore, the proposition 3 follows from the theorem 3.

The Proposition is proved.

Some sufficient conditions for the existence of a proper coloring of edges of a bipartite multigraph with the fixed number of edges colored by each color are found in [9–12].

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Resume (in Armenian)

Let $G = (V_1(G), V_2(G), E(G))$ be a bipartite multigraph, and $R \subseteq V_1(G) \cup V_2(G)$. A proper coloring of edges of G with the colors $1, \dots, t$ is called interval (respectively, continuous) on R , if each color is used for at least one edge and the edges incident with each vertex $x \in R$ are colored by $d(x)$ consecutive colors (respectively, by the colors $1, \dots, d(x)$), where $d(x)$ is a degree of the vertex x . We denote by $w_1(G)$ and $W_1(G)$, respectively, the least and the greatest values of t , for which there exists an interval on $V_1(G)$ coloring of the multigraph G with the colors $1, \dots, t$.

In the paper the following basic results are obtained.

Theorem 2. For an arbitrary k , $w_1(G) \leq k \leq W_1(G)$, there is an interval on $V_1(G)$ coloring of the multigraph G with the colors $1, \dots, k$.

Theorem 3. The problem of recognition of the existence of a continuous on $V_1(G)$ coloring of the multigraph G is NP-complete.

Theorem 4. If for any edge $(x, y) \in E(G)$, where $x \in V_1(G)$, the inequality $d(x) \geq d(y)$ holds then there is a continuous on $V_1(G)$ coloring of the multigraph G .

Theorem 1. If G has no multiple edges and triangles, and there is an interval on $V(G)$ coloring of the graph G with the colors $1, \dots, k$, then $k \leq |V(G)| - 1$.

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